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Spinor and twistor formulations of massless particles with rigidity

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ABSTRACT

The 4-dimensional model of a massless particle with rigidity whose Lagrangian is proportional to its world-line curvature is reformulated in terms of spinor and twistor variables. We begin with a first-order Lagrangian that is equivalent to the original Lagrangian proportional to the extrinsic curvature of a particle world-line. The first-order Lagrangian is written in terms of spacetime and spinor variables, leading to a spinor representation of the Lagrangian. Then its corresponding action is expressed in terms of twistor variables, leading to the gauged Shirafuji action.

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1. Introduction

Several types of 4-dimensional models of relativistic point particles with rigidity have been studied until recently from various points of view [1–16]. The actions of these models contain the extrinsic curvature of a world-line traced out by a particle. By choosing the arc-length, l , of the world-line to be a parameter along it, the actions take the following form:

$$S = \int_{l_0}^{l_1} dl (-m - sK^\gamma), \quad (1)$$

where m , s and γ are real constants, while $K = K(l)$ denotes the extrinsic curvature of the world-line.¹ Each of the models is characterized by a fixed value of γ and according to whether m vanishes or not. The models that have most been explored until now are those with $\gamma = 1$ [1–10]. Among them, the massless particle model specified by $m = 0$ and $\gamma = 1$ is well understood from a physical viewpoint. In fact, Plyushchay demonstrated that this model describes a massless spinning particle of helicity s [5,6]. (Here, s in the action (1) turns out to be replaced by $-|s|$.) Considering this, we come up with an idea that the model of a massless particle with rigidity, defined by the Lagrangian $L_K(l) := -|s|K$, may be expressed in terms of (commutative) spinor and twistor variables, because these variables can be utilized for describing

massless spinning particles of arbitrary helicities; see Ref. [17] and references therein. To realize this idea, in the present Letter, we follow Shirafuji's method used for finding spinor and twistor formulations of massless spinless particles [18].

In order to apply Shirafuji's method to the model of a massless particle with rigidity, we first provide an appropriate first-order Lagrangian and prove that it is indeed equivalent to the Lagrangian L_K . Then we simultaneously solve the constraint equations derived from the first-order Lagrangian by using 2-component Weyl spinors, obtaining a set of solutions. Substituting the solutions into the first-order Lagrangian, we have a Lagrangian written in terms of spacetime and spinor variables. After some manipulations, this Lagrangian reduces to a simpler one investigated in Ref. [17]. In this way, we can derive a simple spinor representation of the Lagrangian for a massless particle with rigidity. Furthermore, from the spinor representation, we find twistor representations of the action for a massless particle with rigidity. It is seen that each of the twistor representations is precisely the one referred to in Refs. [17,19] as the gauged Shirafuji action.

This Letter is organized as follows: Section 2 provides a brief review of Shirafuji's method. A first-order Lagrangian for a massless particle with rigidity is established in Section 3. In Section 4, we derive a spinor representation of the Lagrangian, and in Section 5, we find its corresponding twistor representations of the action. Section 6 is devoted to a summary and discussion.

2. Shirafuji's method for a massless spinless particle

Let $x^\mu = x^\mu(\tau)$ ($\mu = 0, 1, 2, 3$) be spacetime coordinates of a point particle propagating in 4-dimensional Minkowski space, \mathbf{M} , with the metric tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Here, τ ($\tau_0 \leq \tau \leq \tau_1$) is an arbitrary parameter of the world-line of the particle,

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¹ A more general action $S_F = \int_{l_0}^{l_1} dl F(K)$, with F being an arbitrary function of K , was considered by Plyushchay [15]. Similar action defined in D -dimensions was also considered by Nesterenko et al. [16].

being chosen in such a manner that $dx^0/d\tau > 0$. Shirafuji's method begins with the first-order Lagrangian for a massless spinless particle:

$$L_0 = -\dot{x}^\mu p_\mu + \frac{1}{2} e p_\mu p^\mu, \quad (2)$$

where $e = e(\tau)$ and $p_\mu = p_\mu(\tau)$ are understood as real auxiliary fields on the parameter space $\mathcal{T} := \{\tau \mid \tau_0 \leq \tau \leq \tau_1\}$. A dot over a variable denotes its derivative w.r.t. τ .

The Euler–Lagrange equation for e reads $p_\mu p^\mu = 0$, which can easily be solved as $p_{\alpha\dot{\alpha}} = \bar{\pi}_\alpha \pi_{\dot{\alpha}}$ in terms of a (commutative) 2-component spinor $\bar{\pi}_\alpha = \bar{\pi}_\alpha(\tau)$ ($\alpha = 0, 1$) and its complex conjugate $\pi_{\dot{\alpha}} = \pi_{\dot{\alpha}}(\tau)$ ($\dot{\alpha} = \dot{0}, \dot{1}$). Here, the bispinor notation has been used for p_μ .² Substituting $p_{\alpha\dot{\alpha}} = \bar{\pi}_\alpha \pi_{\dot{\alpha}}$ into Eq. (2), we have

$$L_0 = -\dot{x}^{\alpha\dot{\alpha}} \bar{\pi}_\alpha \pi_{\dot{\alpha}}, \quad (3)$$

which is precisely a spinor representation of the Lagrangian for a massless spinless particle [18].

We can also write the Lagrangian (3) as

$$L_0 = \frac{i}{2} (\bar{X}_A \dot{X}^A - X^A \dot{\bar{X}}_A) \quad (4)$$

in terms of the twistor $X^A := (ix^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}}, \pi_{\dot{\alpha}})$ ($A = 0, 1, 2, 3$) and its dual twistor $\bar{X}_A := (\bar{\pi}_\alpha, -ix^{\alpha\dot{\alpha}} \bar{\pi}_\alpha)$. Note that these twistors fulfill the null twistor condition $\bar{X}_A X^A = 0$, which implies that the particle is spinless [18,21,22]. (For a non-null twistor, see Section 5.) In Sections 4 and 5, we will apply Shirafuji's method to the model of a massless particle with rigidity.

3. A first-order Lagrangian for a massless particle with rigidity

Before applying Shirafuji's method, we establish a first-order Lagrangian for a massless particle with rigidity.

Let us consider the action

$$S = \int_{\tau_0}^{\tau_1} d\tau L \quad (5)$$

with the Lagrangian

$$L = (q^\mu - \dot{x}^\mu) p_\mu - (\dot{q}^\mu - b q^\mu) r_\mu + 2a(\pm\sqrt{q^2 r^2} - s), \quad (6)$$

supplemented by the condition $q^2 \neq 0$. Here, $q^2 := q_\mu q^\mu$, $r^2 := r_\mu r^\mu$, and s is a dimensionless real constant (in units such that $\hbar = c = 1$). The variables $q^\mu = q^\mu(\tau)$, $p_\mu = p_\mu(\tau)$, $r_\mu = r_\mu(\tau)$, $a = a(\tau)$, and $b = b(\tau)$ are understood as real fields on the parameter space \mathcal{T} . These fields are assumed to transform under the reparametrization $\tau \rightarrow \tau'(\tau)$ ($d\tau'/d\tau > 0$) as

$$q^\mu(\tau) \rightarrow q'^\mu(\tau') = \frac{d\tau}{d\tau'} q^\mu(\tau), \quad (7)$$

$$p_\mu(\tau) \rightarrow p'_\mu(\tau') = p_\mu(\tau), \quad (8)$$

$$r_\mu(\tau) \rightarrow r'_\mu(\tau') = \frac{d\tau'}{d\tau} r_\mu(\tau), \quad (9)$$

$$a(\tau) \rightarrow a'(\tau') = \frac{d\tau}{d\tau'} a(\tau), \quad (10)$$

$$b(\tau) \rightarrow b'(\tau') = \frac{d\tau}{d\tau'} b(\tau) + \frac{d\tau'}{d\tau} \frac{d^2\tau}{d\tau'^2}. \quad (11)$$

² The bispinor notation $p_{\alpha\dot{\alpha}}$ and the 4-vector notation p_μ are related as follows [20]:

$$\begin{pmatrix} p_{0\dot{0}} & p_{0\dot{1}} \\ p_{1\dot{0}} & p_{1\dot{1}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}.$$

It is easy to see that $p_{\alpha\dot{\alpha}}$ is Hermitian if and only if p_μ is real.

Accordingly, $\nabla q^\mu := \dot{q}^\mu - b q^\mu$, which is included in the Lagrangian (6), transforms homogeneously:

$$\nabla q^\mu(\tau) \rightarrow \nabla' q'^\mu(\tau') = \left(\frac{d\tau}{d\tau'} \right)^2 \nabla q^\mu(\tau). \quad (12)$$

In this transformation, b plays the role of a gauge field. It is obvious that the spacetime coordinates x^μ behave as real scalar fields on \mathcal{T} . Namely, x^μ transform as $x^\mu(\tau) \rightarrow x'^\mu(\tau') = x^\mu(\tau)$. Considering this and Eqs. (7)–(12), we see that the action S remains invariant under the reparametrization. Note that the Lagrangian (6) is first order in \dot{x}^μ and \dot{q}^μ . In this Lagrangian, p_μ , r_μ , a , and b are treated as independent auxiliary fields.

From the Lagrangian (6), the Euler–Lagrange equations for x^μ , q^μ , p_μ , r_μ , a , and b are derived, respectively, as

$$\dot{p}_\mu = 0, \quad (13)$$

$$\dot{r}_\mu + p_\mu + b r_\mu \pm \frac{2ar^2}{\sqrt{q^2 r^2}} q_\mu = 0, \quad (14)$$

$$\dot{x}^\mu - q^\mu = 0, \quad (15)$$

$$\dot{q}^\mu - b q^\mu \mp \frac{2aq^2}{\sqrt{q^2 r^2}} r^\mu = 0, \quad (16)$$

$$\pm\sqrt{q^2 r^2} - s = 0, \quad (17)$$

$$q^\mu r_\mu = 0. \quad (18)$$

Each of Eqs. (13)–(16) includes a derivative term, while Eqs. (17) and (18) include no derivative terms. For this reason, Eqs. (17) and (18) are regarded as constraints. It is seen from Eq. (17) that the sign of s is determined depending on which sign is chosen in $\pm\sqrt{q^2 r^2}$. Taking the derivative of Eq. (17) w.r.t. τ and using Eqs. (14) and (16), we have $q^2 r^\mu p_\mu = 0$. Since $q^2 \neq 0$ has been postulated, it follows that

$$r^\mu p_\mu = 0. \quad (19)$$

Taking the derivative of Eq. (18) w.r.t. τ and using Eqs. (14) and (16), we have

$$q^\mu p_\mu = 0. \quad (20)$$

The derivative of Eq. (20) w.r.t. τ is identically fulfilled with the use of Eqs. (13), (16), (19), and (20), so that no new conditions are derived from Eq. (20). The derivative of Eq. (19) w.r.t. τ , together with Eqs. (13), (14), (19), and (20), yields

$$p^\mu p_\mu = 0. \quad (21)$$

The derivative of Eq. (21) w.r.t. τ gives $p^\mu \dot{p}_\mu = 0$; however, this is identically satisfied with the use of Eq. (13). Hence no new conditions are derived anymore. In addition to Eqs. (17) and (18), Eqs. (19)–(21) are also regarded as constraints.

Using Eq. (16), we can eliminate the auxiliary field r_μ from the Lagrangian (6) to obtain

$$L = (q^\mu - \dot{x}^\mu) p_\mu - 2sa. \quad (22)$$

Here, a is no longer an independent auxiliary field and is determined from Eq. (16) as follows. Contracting Eq. (16) with q_μ and using Eq. (18), we have

$$b = \frac{q\dot{q}}{q^2}, \quad (23)$$

where $q\dot{q} := q_\mu \dot{q}^\mu$. Then Eq. (16) leads to

$$a^2 = \frac{1}{4q^2} (\dot{q}_\mu - bq_\mu)(\dot{q}^\mu - bq^\mu) = \frac{\dot{q}_\perp^2}{4q^2} \quad (24)$$

with $\dot{q}_\perp^\mu := \dot{q}^\mu - q^\mu(q\dot{q})/q^2$. In this way, the real field a is determined to be $a = \pm(1/2)\sqrt{\dot{q}_\perp^2/q^2}$, from which we see that there are two possibilities: (a) $q^2 > 0$, $\dot{q}_\perp^2 \geq 0$, and (b) $q^2 < 0$, $\dot{q}_\perp^2 \leq 0$. In both cases, $q^2\dot{q}^2 \geq (q\dot{q})^2$ is valid, as can be shown using $q^2\dot{q}_\perp^2 = q^2\dot{q}^2 - (q\dot{q})^2$. On the other hand, when q^μ is timelike (i.e., $q^2 > 0$), it can be shown that $q^2\dot{q}^2 \leq (q\dot{q})^2$.³ Hence, the case (a) involves $q^2\dot{q}^2 \geq (q\dot{q})^2$ and $q^2\dot{q}^2 \leq (q\dot{q})^2$ simultaneously, and consequently $q^2\dot{q}^2 = (q\dot{q})^2$ is required for any $\tau \in \mathcal{T}$ as a purely mathematical condition, without the use of equations of motion. However, this equality implies $\dot{q}_\perp^2 = 0$, leading to a meaningless situation in which $S = 0$ holds off shell. Therefore it turns out that the case (a) is forbidden and only the case (b) is allowed. In the case (b), the condition $r^2 < 0$ is required so that the Lagrangian (6) can be a real function.

The Lagrangian (22) can be written explicitly as

$$L = (q^\mu - \dot{x}^\mu)p_\mu \mp s\sqrt{\frac{\dot{q}_\perp^2}{q^2}}, \quad (25)$$

which becomes

$$L = \mp s\sqrt{\frac{\dot{x}_\perp^2}{\dot{x}^2}} = \mp s\frac{\sqrt{\dot{x}^2\dot{x}^2 - (\dot{x}\dot{x})^2}}{-\dot{x}^2} \quad (26)$$

by eliminating q^μ and p_μ with the use of Eq. (15). In Eq. (26), $\dot{x}^2 < 0$ and $\dot{x}_\perp^2 \leq 0$ are taken into account in accordance with the conditions in the case (b). Also, in Eqs. (25) and (26), one of the signs in the symbol \mp is chosen so that the Lagrangian can be negative definite whether s is positive or negative. As a result, $\mp s = -|s|$ holds and the Lagrangian (26) reads

$$L = -|s|\sqrt{\frac{\dot{x}_\perp^2}{\dot{x}^2}}. \quad (27)$$

This is exactly the Lagrangian for a massless particle with rigidity evaluated as a function of τ : $L(\tau) = \sqrt{-\dot{x}^2}L_K = -|s|\sqrt{-\dot{x}^2}K$ [4–7]. (Note that $\mp s = -|s|$ is compatible with Eq. (17), leading to a consistent result $\sqrt{q^2r^2} = |s|$.) The condition $\dot{x}^2 < 0$ implies that the particle moves at a speed faster than the speed of light. This motion can be understood as a classical analog of the Zitterbewegung of a massless spinning particle [23]. (The authors thank M.S. Plyushchay for telling us this point.) As expected, the reparametrization invariance of the action $S = \int_{\tau_0}^{\tau_1} d\tau L$ is maintained with the Lagrangian (27). In our approach, the Lagrangian (27) has been obtained from the Lagrangian (6) by eliminating the auxiliary fields p_μ , r_μ , a , and b , and furthermore the field q^μ . For this reason, the Lagrangian (6) is established as a first-order Lagrangian for a massless particle with rigidity, being considered the fact that the Lagrangian (6) is first order in \dot{x}^μ and \dot{q}^μ .

4. Spinor representation of the Lagrangian

In this section, we derive a spinor representation of the Lagrangian (27) by following the procedure for deriving the spinorial Lagrangian (3) from the first-order Lagrangian (2).

³ Let u^μ be a timelike vector in \mathbf{M} , satisfying $u^2 := u_\mu u^\mu > 0$, and v^μ an arbitrary vector in \mathbf{M} . Since u^μ is timelike, we can choose the rest frame such that $u^i = 0$ ($i = 1, 2, 3$). In this frame, $u^2 = u_\mu u^\mu$ and $uv = u_\mu v^\mu$ reduce to $u^2 = (u^0)^2$ and $uv = u^0v^0$, respectively. Then we can easily show that $u^2v^2 - (uv)^2 = -(u^0)^2 \sum_{i=1}^3 v^i v^i \leq 0$. Because u^2 , v^2 , and uv are Lorentz scalars, $u^2v^2 \leq (uv)^2$ is valid in arbitrary reference frames.

The first-order Lagrangian (6) can be expressed in bispinor notation [20] as

$$L = (q^{\alpha\dot{\alpha}} - \dot{x}^{\alpha\dot{\alpha}})p_{\alpha\dot{\alpha}} - (\dot{q}^{\alpha\dot{\alpha}} - bq^{\alpha\dot{\alpha}})r_{\alpha\dot{\alpha}} + 2a(\pm\sqrt{q^2r^2} - s), \quad (28)$$

where $q^2 = q_{\alpha\dot{\alpha}}q^{\alpha\dot{\alpha}}$ and $r^2 = r_{\alpha\dot{\alpha}}r^{\alpha\dot{\alpha}}$ ($\alpha = 0, 1$; $\dot{\alpha} = \dot{0}, \dot{1}$). The constraints (21), (20), (19), and (18) can be written, respectively, as

$$p^{\alpha\dot{\alpha}}p_{\alpha\dot{\alpha}} = 0, \quad (29)$$

$$q^{\alpha\dot{\alpha}}p_{\alpha\dot{\alpha}} = 0, \quad (30)$$

$$r^{\alpha\dot{\alpha}}p_{\alpha\dot{\alpha}} = 0, \quad (31)$$

$$q^{\alpha\dot{\alpha}}r_{\alpha\dot{\alpha}} = 0. \quad (32)$$

These constraint equations can be solved simultaneously in terms of (commutative) 2-component spinors $\bar{\omega}_\alpha = \bar{\omega}_\alpha(\tau)$ and $\chi^\alpha = \chi^\alpha(\tau)$ and their complex conjugates $\bar{\omega}_{\dot{\alpha}} = \bar{\omega}_{\dot{\alpha}}(\tau)$ and $\bar{\chi}^{\dot{\alpha}} = \bar{\chi}^{\dot{\alpha}}(\tau)$:

$$p_{\alpha\dot{\alpha}} = \bar{\omega}_\alpha \bar{\omega}_{\dot{\alpha}}, \quad (33)$$

$$q^{\alpha\dot{\alpha}} = f(\bar{\omega}^\alpha \bar{\chi}^{\dot{\alpha}} + \chi^\alpha \bar{\omega}^{\dot{\alpha}}), \quad (34)$$

$$r_{\alpha\dot{\alpha}} = ig(\bar{\omega}_\alpha \bar{\chi}^{\dot{\alpha}} - \chi_\alpha \bar{\omega}^{\dot{\alpha}}), \quad (35)$$

where $f = f(\tau)$ and $g = g(\tau)$ are real functions on \mathcal{T} . Using the conventional formula $\iota^\alpha \kappa_\alpha = \epsilon^{\alpha\beta} \iota_\beta \kappa_\alpha = \iota^\alpha \kappa^\beta \epsilon_{\beta\alpha}$ valid for arbitrary undotted spinors ι_α and κ_α , together with its complex conjugate valid for $\bar{\iota}_{\dot{\alpha}}$ and $\bar{\kappa}_{\dot{\alpha}}$, we can actually verify that Eqs. (33)–(35) simultaneously satisfy all the constraints (29)–(32). (Here, $\epsilon^{\alpha\beta}$ and $\epsilon_{\beta\alpha}$ denote Levi-Civita symbols specified by $\epsilon^{01} = \epsilon_{01} = 1$.) We assume that $\bar{\omega}_\alpha$, $\bar{\omega}_{\dot{\alpha}}$, χ^α , and $\bar{\chi}^{\dot{\alpha}}$ behave as complex scalar fields on \mathcal{T} , while f and g transform under the reparametrization as

$$f(\tau) \rightarrow f'(\tau') = \frac{d\tau}{d\tau'} f(\tau), \quad (36)$$

$$g(\tau) \rightarrow g'(\tau') = \frac{d\tau'}{d\tau} g(\tau). \quad (37)$$

Then it follows that Eqs. (33), (34), and (35) are compatible with the transformation rules (8), (7), and (9), respectively.

From Eqs. (34) and (35), we can obtain

$$q^2 = -2f^2 |\chi^\alpha \bar{\omega}_\alpha|^2, \quad (38)$$

$$r^2 = -2g^2 |\chi^\alpha \bar{\omega}_\alpha|^2. \quad (39)$$

Since $q^2 \neq 0$, the spinor variables $\bar{\omega}_\alpha$ and χ_α have to be linearly independent. It should be emphasized here that the conditions $q^2 < 0$ and $r^2 < 0$ necessary for the case (b) are automatically fulfilled with Eqs. (38) and (39). Substituting Eqs. (33), (34), (35), (38), and (39) into Eq. (28), we have

$$L = -\dot{x}^{\alpha\dot{\alpha}} \bar{\omega}_\alpha \bar{\omega}_{\dot{\alpha}} - ifg(\bar{\chi}^{\dot{\alpha}} \bar{\omega}_\alpha \chi^\alpha \dot{\bar{\omega}}_\alpha - \chi^\alpha \bar{\omega}_\alpha \bar{\chi}^{\dot{\alpha}} \dot{\bar{\omega}}_\alpha - \bar{\chi}^{\dot{\alpha}} \bar{\omega}_\alpha \dot{\chi}^\alpha \bar{\omega}_\alpha + \chi^\alpha \bar{\omega}_\alpha \dot{\bar{\chi}}^{\dot{\alpha}} \bar{\omega}_\alpha) + 2a(\pm 2|fg| |\chi^\alpha \bar{\omega}_\alpha \bar{\chi}^{\dot{\alpha}} \bar{\omega}_\alpha - s). \quad (40)$$

Because $\chi^\alpha \bar{\omega}_\alpha$ is a complex-valued function of τ , it can be expressed as $Re^{i\Theta}$, with $R(\tau) := |\chi^\alpha \bar{\omega}_\alpha|$ and $\Theta(\tau) := \arg(\chi^\alpha \bar{\omega}_\alpha)$. Now we set the following condition between f and g :

$$fg = \pm \frac{1}{2R} = \frac{\exp\{i\varphi(\pm)\}}{2R}, \quad (41)$$

where $\varphi(+):=0$ and $\varphi(-):=\pi$. This condition is compatible with the transformation rules (36) and (37). Substituting Eq. (41) into Eq. (40), we can write L as

$$L = -\dot{\chi}^{\alpha\dot{\alpha}} \bar{\omega}_{\alpha} \bar{\omega}_{\dot{\alpha}} - \frac{i}{2} (e^{-i\Phi(\pm)} \chi^{\alpha} \dot{\bar{\omega}}_{\alpha} - e^{i\Phi(\pm)} \bar{\chi}^{\dot{\alpha}} \dot{\omega}_{\dot{\alpha}} - e^{-i\Phi(\pm)} \dot{\chi}^{\alpha} \bar{\omega}_{\alpha} + e^{i\Phi(\pm)} \dot{\bar{\chi}}^{\dot{\alpha}} \omega_{\dot{\alpha}}) + a(e^{-i\Phi(\pm)} \chi^{\alpha} \bar{\omega}_{\alpha} + e^{i\Phi(\pm)} \bar{\chi}^{\dot{\alpha}} \omega_{\dot{\alpha}} - 2s), \quad (42)$$

where $\Phi(\pm) := \Theta - \varphi(\pm)$.

In terms of the new spinor variables

$$\begin{aligned} \bar{\pi}_{\alpha} &:= e^{-i\Phi(\pm)/2} \bar{\omega}_{\alpha}, & \pi_{\dot{\alpha}} &:= e^{i\Phi(\pm)/2} \omega_{\dot{\alpha}}, \\ \psi^{\alpha} &:= e^{-i\Phi(\pm)/2} \chi^{\alpha}, & \bar{\psi}^{\dot{\alpha}} &:= e^{i\Phi(\pm)/2} \bar{\chi}^{\dot{\alpha}}, \end{aligned} \quad (43)$$

Eq. (42) can simply be expressed as

$$L = -\dot{\chi}^{\alpha\dot{\alpha}} \bar{\pi}_{\alpha} \pi_{\dot{\alpha}} - \frac{i}{2} (\psi^{\alpha} \dot{\bar{\pi}}_{\alpha} - \bar{\psi}^{\dot{\alpha}} \dot{\pi}_{\dot{\alpha}} - \dot{\psi}^{\alpha} \bar{\pi}_{\alpha} + \dot{\bar{\psi}}^{\dot{\alpha}} \pi_{\dot{\alpha}}) + a(\psi^{\alpha} \bar{\pi}_{\alpha} + \bar{\psi}^{\dot{\alpha}} \pi_{\dot{\alpha}} - 2s). \quad (44)$$

In the process of deriving Eq. (44) from Eq. (42), four terms containing $d\Phi(\pm)/d\tau$ appear in L ; however, they cancel out each other, and consequently $d\Phi(\pm)/d\tau$ does not remain in Eq. (44). Eqs. (33)–(35) can be written, in terms of the new spinor variables, as

$$p_{\alpha\dot{\alpha}} = \bar{\pi}_{\alpha} \pi_{\dot{\alpha}}, \quad (45)$$

$$q^{\alpha\dot{\alpha}} = f(\bar{\pi}_{\alpha} \bar{\psi}^{\dot{\alpha}} + \psi^{\alpha} \pi_{\dot{\alpha}}), \quad (46)$$

$$r_{\alpha\dot{\alpha}} = ig(\bar{\pi}_{\alpha} \bar{\psi}^{\dot{\alpha}} - \psi^{\alpha} \pi_{\dot{\alpha}}). \quad (47)$$

Here, fg is given by $fg = \pm(1/2)|\psi^{\alpha} \bar{\pi}_{\alpha}|^{-1}$. We thus see that the forms of the solutions (33)–(35) do not change after rewriting them in terms of the new spinor variables. As we have seen, the Lagrangian (44) as well as the Lagrangian (27) is originally obtained from the Lagrangian (6) by classical-mechanical treatments. Therefore the equivalence between the Lagrangians (44) and (27) is established at least at the classical level, and the Lagrangian (44) can now be considered a spinor representation of the Lagrangian for a massless particle with rigidity. The reparametrization invariance of the action $S = \int_{\tau_0}^{\tau_1} d\tau L$ can be seen immediately with the Lagrangian (44).

Subtracting the total derivative term $(i/2)d(\psi^{\alpha} \bar{\pi}_{\alpha} - \bar{\psi}^{\dot{\alpha}} \pi_{\dot{\alpha}})/d\tau$ from Eq. (44), we have a simpler but equivalent Lagrangian

$$\tilde{L} := -\dot{\chi}^{\alpha\dot{\alpha}} \bar{\pi}_{\alpha} \pi_{\dot{\alpha}} - i(\psi^{\alpha} \dot{\bar{\pi}}_{\alpha} - \bar{\psi}^{\dot{\alpha}} \dot{\pi}_{\dot{\alpha}}) + a(\psi^{\alpha} \bar{\pi}_{\alpha} + \bar{\psi}^{\dot{\alpha}} \pi_{\dot{\alpha}} - 2s), \quad (48)$$

which is precisely what the authors have investigated in Ref. [17]. As stated therein, the Lagrangian (48) describes a free massless spinning particle of helicity s propagating in \mathbf{M} . This leads to the fact that the equivalent Lagrangian (27) also describes a free massless spinning particle of helicity s . According to the canonical formalism based on the Lagrangian (48) and the subsequent canonical quantization, the possible values of s are restricted to either integer or half-integer values owing to the fact that meaningful wave functions in \mathbf{M} have to be Lorentz spinors [17]. The values of s derived in this manner are identical with those obtained in a quantization procedure of the classical system governed by the Lagrangian (27) [6].

5. Twistor representation of the action

In this section, we find twistor representations of the action with the Lagrangian (27) by following the way of finding the twistorial Lagrangian (4).

We first define a 2-component spinor ω^{α} and its complex conjugate $\bar{\omega}^{\dot{\alpha}}$ by

$$\omega^{\alpha} := ix^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}} + \psi^{\alpha}, \quad \bar{\omega}^{\dot{\alpha}} := -ix^{\alpha\dot{\alpha}} \bar{\pi}_{\alpha} + \bar{\psi}^{\dot{\alpha}}. \quad (49)$$

Then, in terms of ω^{α} , $\bar{\omega}^{\dot{\alpha}}$, $\pi_{\dot{\alpha}}$, and $\bar{\pi}_{\alpha}$, the Lagrangian (44) can be written as

$$L = \frac{i}{2} (\bar{\pi}_{\alpha} \dot{\omega}^{\alpha} + \bar{\omega}^{\dot{\alpha}} \dot{\pi}_{\dot{\alpha}} - \omega^{\alpha} \dot{\bar{\pi}}_{\alpha} - \pi_{\dot{\alpha}} \dot{\bar{\omega}}^{\dot{\alpha}}) + a(\bar{\pi}_{\alpha} \omega^{\alpha} + \bar{\omega}^{\dot{\alpha}} \pi_{\dot{\alpha}} - 2s). \quad (50)$$

Note here that the spacetime variables $x^{\alpha\dot{\alpha}}$ no longer appear apparently in this expression. We can concisely express the action with the Lagrangian (50) in terms of the twistor $Z^A := (\omega^{\alpha}, \pi_{\dot{\alpha}})$ ($A = 0, 1, 2, 3$) and its dual twistor $\bar{Z}_A := (\bar{\pi}_{\alpha}, \bar{\omega}^{\dot{\alpha}})$:

$$\begin{aligned} S &= \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A \dot{Z}^A - Z^A \dot{\bar{Z}}_A) + a(\bar{Z}_A Z^A - 2s) \right] \\ &= \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} (\bar{Z}_A D Z^A - Z^A \bar{D} \bar{Z}_A) - 2sa \right], \end{aligned} \quad (51)$$

where $D := d/d\tau - ia$. This action is precisely the one referred to in Ref. [19] as the gauged Shirafuji action. It remains invariant under the local $U(1)$ transformation

$$Z^A \rightarrow Z'^A = \exp\{i\theta(\tau)\} Z^A, \quad (52)$$

$$\bar{Z}_A \rightarrow \bar{Z}'_A = \exp\{-i\theta(\tau)\} \bar{Z}_A, \quad (53)$$

$$a \rightarrow a' = a + \frac{d\theta(\tau)}{d\tau}, \quad (54)$$

where θ is a real gauge function of τ . In Eq. (51), $\bar{Z}_A D Z^A$ and $Z^A \bar{D} \bar{Z}_A$ themselves are gauge invariant, while the gauge invariance of $\int_{\tau_0}^{\tau_1} d\tau a$ is ensured by imposing an appropriate boundary condition such as $\theta(\tau_1) = \theta(\tau_0)$. It should be noted that the field a introduced in Eq. (6) as an auxiliary field is now realized as a $U(1)$ gauge field on \mathcal{T} . Variation of the action (51) w.r.t. a yields $\bar{Z}_A Z^A = 2s$, from which we see that Z^A is non-null if the particle is spinful.

Next, we carry out the scaling $Z^A \rightarrow \sqrt{\lambda} Z^A$ and $\bar{Z}_A \rightarrow \sqrt{\lambda} \bar{Z}_A$, with $\lambda = \lambda(\tau)$ being a positive scalar field on \mathcal{T} . Thereby, Eq. (51) becomes

$$S = \int_{\tau_0}^{\tau_1} d\tau \left[\frac{i}{2} \lambda (\bar{Z}_A D Z^A - Z^A \bar{D} \bar{Z}_A) - 2sa \right]. \quad (55)$$

This action remains invariant under the complexified local scale transformation or, in other words, a combination of the local scale and local $U(1)$ transformations:

$$Z^A \rightarrow Z'^A = \exp\{\vartheta(\tau) + i\theta(\tau)\} Z^A, \quad (56)$$

$$\bar{Z}_A \rightarrow \bar{Z}'_A = \exp\{\vartheta(\tau) - i\theta(\tau)\} \bar{Z}_A, \quad (57)$$

$$a \rightarrow a' = a + \frac{d\theta(\tau)}{d\tau}, \quad (58)$$

$$\lambda \rightarrow \lambda' = \exp\{-2\vartheta(\tau)\} \lambda, \quad (59)$$

where ϑ and θ are real gauge functions of τ [17,19]. The action (51) can be regarded as the action (55) in a particular gauge $\lambda = 1$. The fact that the action (55) remains invariant under the complexified local scale transformation implies that the action (55) is really defined for the proportionality class called the projective twistor, $[Z^A] := \{cZ^A \mid c \in \mathbb{C} \setminus \{0\}\}$. The action (55) is thus considered to be described with the projective twistor $[Z^A]$. This statement is consistent with the fact that in twistor theory, projective twistors are more essential than twistors themselves.

In this Letter, the actions (51) and (55) have eventually been found from the action (5) via the spinorial Lagrangian (44). For this reason, the actions (51) and (55) can be treated as twistor representations of the action for a massless particle with rigidity. The canonical quantization of the classical system governed by the action (51) turns out to be the so-called twistor quantization [21, 22, 24]. In the twistor quantization procedure, the possible values of s are restricted again to either integer or half-integer values by imposing the single-valuedness condition on twistor (wave) functions.

6. Summary and discussion

In this Letter, we have found spinor and twistor representations of the Lagrangian (or action) for a massless particle with rigidity on the basis of the first-order Lagrangian (6) by following Shirafuji's method. We first verified that the Lagrangian (6) is indeed equivalent to the original Lagrangian (27). Then we demonstrated that the Lagrangian (6) can be expressed as the spinorial Lagrangian (44) and furthermore leads to the twistorial actions (51) and (55). These spinorial and twistorial forms are thus established as representations of the Lagrangian (27) or its corresponding action. With these representations, referring to, e.g., Refs. [17, 24], we saw that the allowed values of s are restricted upon quantization to either integer or half-integer values. This is consistent with results obtained in Ref. [6]. It is worth mentioning here that unlike the original Lagrangian (27), the Lagrangians (6) and (44) are applicable to the spinless case $s = 0$.

We have shown the equivalence between the Lagrangians (27) and (44) by using classical equations of motion. To make sure of this equivalence more closely, it is necessary to establish the equivalence at the quantum-theoretical level. Also, it would be of considerable interest to study spinor and twistor formulations of massive particles with rigidity. In addition, the 3-dimensional model of a relativistic particle with torsion [25] should also be treated in the spinor and twistor approaches, although this model involves complicated constraints. We hope to tackle these issues in the future.

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